MATH 2060 TUTOS

Example 1. Let $f(x) = \frac{1}{x^3}$ on $[a,b]$, where $0 < a < b$. Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. Find tags points $t_i \in [x_{i-1}, x_i]$ explicitly in terms of x_{i-1}, x_i such that the corresponding tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ satisfies

$$S(f; \dot{\mathcal{P}}) = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right).$$

Hence show that $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \int_a^b \frac{1}{x^3}\, dx = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$.

Ans: Consider the trivial partition $a < b$ with tag pt. c.

Want: $f(c)(b-a) = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

$$\frac{1}{c^3}(b-a) = \frac{b^2 - a^2}{2a^2b^2}$$

$$a^3 = \frac{2a^3b}{2b} < c^3 = \frac{2a^2b^2}{b+a} < \frac{2ab^3}{2a} = b^3$$

So $c = \left(\frac{2a^2b^2}{a+b}\right)^{1/3} \in (a,b)$

This leads us to set

$$t_i := \left(\frac{2x_{i-1}^2 x_i^2}{x_{i-1} + x_i}\right)^{1/3} \quad \text{for } i = 1, \dots, n.$$

Then $x_{i-1} < t_i < x_i$

So $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ satisfies

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2}\left(\frac{1}{x_{i-1}^2} - \frac{1}{x_i^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

Now let $\dot{Q} = \{[x_{i-1}, x_i], q_i\}_{i=1}^n$ be an arbitrary tagged partition of $[a,b]$.
with $\|\dot{Q}\| < \delta$.
Define $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ as above
Then

$$|f(t_i) - f(q_i)| = \left|\frac{1}{t_i^3} - \frac{1}{q_i^3}\right|$$
$$= |q_i - t_i| \frac{|q_i^2 + q_i t_i + t_i^2|}{t_i^3 q_i^3}$$
$$< \underline{\delta} \left(\frac{3b^2}{a^6}\right) \qquad \text{since} \quad t_i, q_i \in [x_{i-1}, x_i].$$

Hence

$$|S(f;\dot{P}) - S(f;\dot{Q})| = \left|\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n f(q_i)(x_i - x_{i-1})\right|$$
$$\leq \sum_{i=1}^n |f(t_i) - f(q_i)|(x_i - x_{i-1})$$
$$< \delta\left(\frac{3b^2}{a^6}\right)\sum_{i=1}^n (x_i - x_{i-1})$$
$$= \delta\left(\frac{3b^2}{a^6}\right)(b-a)$$

Thus, $\forall\, \varepsilon > 0$, choose $\delta_\varepsilon := \varepsilon / \left[\left(\frac{3b^2}{a^6}\right)(b-a)\right]$,
so that if $\|\dot{Q}\| < \delta_\varepsilon$, then

$$\left|S(f;\dot{Q}) - \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right| < \varepsilon$$

Therefore $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ //

Thm 7.2.1 (Cauchy Criterion)

A fcn $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$ iff

$\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t.

if $\dot{P}$, $\dot{Q}$ are tagged partitions of $[a,b]$ with $\|\dot{P}\|, \|\dot{Q}\| < \eta_\varepsilon$,

then
$$\left| S(f;\dot{P}) - S(f;\dot{Q}) \right| < \varepsilon$$

Thm 7.2.3 (Squeeze Thm)

Let $f : [a,b] \to \mathbb{R}$. Then $f \in \mathcal{R}[a,b]$ iff

$\forall \varepsilon > 0$, $\exists$ fcns $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a,b]$

with $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ $\forall x \in [a,b]$

s.t.
$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

8. Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.

8) Ans: Suppose $f(x_0) > 0$ $\exists\, x_0 \in [a,b]$

WLOG, we may assume $x_0 \in (a,b)$

Then $\exists\, \delta > 0$ s.t.

$$f(x) > m := f(x_0)/2 \qquad \forall\, x \in (x_0-\delta, x_0+\delta) \cap [a,b].$$

By choosing $\delta > 0$ small enough, we may assume that

$$(x_0-\delta, x_0+\delta) \subseteq [a,b].$$

Consider the step fcn φ on $[a,b]$ defined by

$$\varphi(x) := \begin{cases} m & , x \in (x_0-\delta, x_0+\delta) \\ 0 & , \text{elsewhere} \end{cases}$$

Then $f, \varphi \in \mathcal{R}[a,b]$ and $f(x) \geq \varphi(x)$ $\forall x \in [a,b]$.

By Thm 7.1.5 c),

$$\begin{aligned} \int_a^b f &\geq \int_a^b \varphi \\ &= m(2\delta) \quad \text{by Lemma 7.2.4} \\ &> 0, \end{aligned}$$

contradicting the assumption that $\int_a^b f = 0$

Hence $f(x) = 0$ $\forall\, x \in [a,b]$

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8. Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.

9) Ans: Let $h : [0,1] \to \mathbb{R}$ be given by

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

$\leftarrow$ not identically zero.

Clearly h is discontinuous at 0.

Next we show that $h \in \mathcal{R}[0,1]$ using Squeeze Thm.

Let $\varepsilon \in (0,1)$.

Let α_ε be the constant 0 fcn, $\alpha_\varepsilon(x) = 0$ on $[0,1]$

Let ω_ε be the step fcn:

$$\omega_\varepsilon(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

Then $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[0,1]$ with

$$\alpha_\varepsilon(x) \leq h(x) \leq \omega_\varepsilon(x) \quad \forall\, x \in [0,1].$$

and s.t.

$$\int_0^1 (\omega_\varepsilon - \alpha_\varepsilon) = \int_0^1 \omega_\varepsilon$$

$$= 1\,(\varepsilon/2 - 0) = \varepsilon/2 < \varepsilon.$$

By Squeeze Thm 7.2.3, $h \in \mathcal{R}[0,1]$.

Furthermore, by Thm 7.1.5

$$0 = \int_0^1 \alpha_\varepsilon \leq \int_0^1 h \leq \int_0^1 \omega_\varepsilon = \varepsilon/2$$

Letting $\varepsilon \to 0^+$, we have $\int_0^1 h = 0$.

Example

Let a function $f\colon [0, \frac{\pi}{2}] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos^2 x, & x \in [0, \frac{\pi}{2}] \cap \mathbb{Q} \\ 0, & \text{else.} \end{cases}$$

Is this function Riemann integrable?

Ans: Try to apply Cauchy Criterion to show that $f \notin \mathcal{R}[0, \frac{\pi}{2}]$.

Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[0, \frac{\pi}{2}]$,

and $\dot{\mathcal{P}}_1 = \{[x_{i-1}, x_i], q_i\}_{i=1}^n$ be a tagged partition with tag pts $q_i \in [x_{i-1}, x_i] \cap \mathbb{Q}$

$\dot{\mathcal{P}}_2 = \{[x_{i-1}, x_i], r_i\}_{i=1}^n$ be a tagged partition with tag pts $r_i \in [x_{i-1}, x_i] \setminus \mathbb{Q}$

Then

$$\begin{aligned} S(f; \dot{\mathcal{P}}_1) &= \sum_{i=1}^n f(q_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n \cos^2(q_i)(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{k-1} \frac{1}{2}(x_i - x_{i-1}) \qquad \text{assume } x_{k-1} < \tfrac{\pi}{4} \leq x_k \\ &= \frac{1}{2}(x_{k-1} - x_0) \\ &= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(\frac{\pi}{4} - x_{k-1}\right) \\ &\geq \frac{\pi}{8} - \frac{1}{2}\|\dot{\mathcal{P}}_1\| \end{aligned}$$

and

$$S(f; \dot{\mathcal{P}}_2) = \sum_{i=1}^n \overset{0}{\cancel{f(r_i)}}(x_i - x_{i-1}) = 0$$

(say, uniform partition)

For $\varepsilon_0 = \frac{\pi}{32}$, $\forall \eta > 0$, we can choose $\mathcal{P}$ with $\|\mathcal{P}\| < \min\{\eta, \frac{\pi}{8}\}$

Then $\exists$ corresponding partitions $\dot{\mathcal{P}}_1$, $\dot{\mathcal{P}}_2$ satisfying

$$\|\dot{\mathcal{P}}_1\| = \|\dot{\mathcal{P}}_2\| = \|\mathcal{P}\| < \eta$$

s.t. $$|S(f; \dot{\mathcal{P}}_1) - S(f; \dot{\mathcal{P}}_2)| \geq \frac{\pi}{8} - \frac{1}{2}\left(\frac{\pi}{8}\right) = \frac{\pi}{16} > \varepsilon_0.$$

By Cauchy Criterion 7.2.1, $f \notin \mathcal{R}[0, \frac{\pi}{2}]$ //

16. If f is continuous on $[a, b]$, $a < b$, show that there exists $c \in [a, b]$ such that we have $\int_a^b f = f(c)(b - a)$. This result is sometimes called the *Mean Value Theorem for Integrals*.

Ans: Note that f is cts on $[a,b]$.

By Extreme Value Thm, $\exists\, x_1, x_2 \in [a,b]$ s.t.

$$f(x_1) \leq f(x) \leq f(x_2) \qquad \forall x \in [a,b]$$

Note $f \in \mathcal{R}[a,b]$ since it is cts, by Thm 7.2.7.

By Thm 7.1.5,

$$\underbrace{\int_a^b f(x_1)}_{= f(x_1)(b-a)} \leq \int_a^b f \leq \underbrace{\int_a^b f(x_2)}_{= f(x_2)(b-a)}$$

$$\Rightarrow \qquad f(x_1) \leq \frac{\int_a^b f}{b-a} \leq f(x_2)$$

Now, by Intermediate Value Thm,

$\exists\, c$ between x_1, x_2 $\Rightarrow$ $c \in [a,b]$

s.t.
$$f(c) = \frac{\int_a^b f}{b-a}$$

i.e. $$\int_a^b f = f(c)(b-a)$$

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